ON THE CONVECTIVE INSTABILITY OF A LIQUID IN AN INCLINED LAYER OF A POROUS MEDIUM

A. K. Kolesnikov and D. V. Lyubimov UDC 536.25 +532.546

In this paper we study the stability of the equilibrium of a liquid heated from below, wherein the liquid saturates a planar layer of a porous medium arbitrarily inclined to the direction of gravity. We consider the cases for which the boundaries of the layer are heat-conducting and also thermally insulated. In a horizontal layer with heat-conducting boundaries equilibrium is destroyed by perturbations of cellular structure [1]. In a vertical layer the minimum critical temperature gradient corresponds to perturbations of plane-parallel structure. The transition to cellular perturbations in the case of heat-conducting boundaries takes place at an arbitrarily small angle of inclination of the layer to the vertical. For the thermally insulated layer the crisis of equilibrium is connected with plane-parallel perturbations at all angles of inclination.

A two-dimensional infinite layer of a porous medium of thickness 2h, bounded by planes impermeable to a liquid, is inclined at an angle α to the vertical. The conditions of heating are such that an equilibrium state is possible for which a constant vertical temperature gradient is created in the layer.

The heat convection equations for a porous medium have the form [2]

$$\rho_l\left[\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{m}(\mathbf{v}\nabla)\mathbf{v}\right]=-\nabla p+\rho_l g\beta T\boldsymbol{\gamma}-\rho_l\frac{\nu}{K}\mathbf{v}$$
$$(\rho c_p)_s\frac{\partial T}{\partial t}+(\rho c_p)_l\,\mathbf{v}\nabla T=\varkappa_s\Delta T \tag{1}$$
$$\operatorname{div}\mathbf{v}=0$$

where v is the filtration rate, p the convective addition to the pressure, T the temperature, reckoned from a mean value, ρ the density, ν the kinematic viscosity, β the coefficient of volume expansion of the liquid, K the permeability, m the porosity, c_p the specific heat at constant pressure, $\varkappa$ the coefficient of thermal conductivity, g the gravitational acceleration, and γ a unit vector, directed vertically upwards. Quantities provided with the subscripts l and s refer, respectively, to the liquid and to the porous medium saturated by the liquid.

At equilibrium we have

$$\mathbf{v}_0=0,\qquad \nabla T_0=-A\boldsymbol{\gamma},\qquad \nabla p_0=\rho_l g\beta T_0\boldsymbol{\gamma} \tag{2}$$

Here A is a constant equilibrium temperature gradient.

For sufficiently large A the equilibrium becomes unstable, small perturbations increasing with time.

If we use the notation v, T, and p for perturbations of the filtration rate, the temperature, and the pressure, then, taking the Eqs. (2) into account and linearizing, we have

$$\rho_l\frac{\partial \mathbf{v}}{\partial t}=-\nabla p+\rho_l g\beta T\boldsymbol{\gamma}-\rho_l\frac{\nu}{K}\mathbf{v}$$
$$(\rho c_p)_s\frac{\partial T}{\partial t}=\varkappa_s\Delta T+(\rho c_p)_l A(\mathbf{v}\boldsymbol{\gamma}) \tag{3}$$
$$\operatorname{div}\mathbf{v}=0$$

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On the layer boundaries the normal component of the filtration velocity vanishes (impermeable boundaries); the perturbations of the temperature for infinitely heat-conducting boundaries and the perturbations of the heat flow for thermally insulated boundaries also vanish at the layer boundaries. The boundary conditions for the system (3) have the form

$$v_x = 0, \quad T = 0 \ \text{ or } \ \partial T/\partial x = 0 \ \text{ for } \ x = \pm h \tag{4}$$

The equations (3) have solutions proportional to $\exp(\lambda t)$ (normal perturbations). We can prove, in a manner analogous to that employed in considering convection in a viscous liquid [3], that for heat applied from below ($A > 0$) the decrements are real and the perturbations vary monotonically with the time. In an isothermal system ($A = 0$) the decrements are negative (stability). The stability boundary is determined from the condition $\lambda = 0$, and the neutral perturbations are obtained from the stationary equations.

We rewrite the equations in dimensionless form, choosing as the units of distance, temperature, velocity, and pressure the respective quantities h, Ah,

$$(Kg\beta A\chi\nu^{-1})^{1/2}, \ (g\beta\rho_l^2 Ah^2\chi\nu K^{-1})^{1/2}; \text{where } \chi \equiv \varkappa_s / (\rho c_p)_l$$

Then

$$\begin{gathered} -\nabla p - \mathbf{v} + CT\boldsymbol{\gamma} = 0 \\ \Delta T + C(\mathbf{v}\boldsymbol{\gamma}) = 0 \\ \operatorname{div} \mathbf{v} = 0 \end{gathered} \tag{5}$$

The boundary conditions are

$$v_x = 0, \quad T = 0 \quad \text{or} \quad \partial T / \partial x = 0 \quad \text{for} \quad x = \pm 1 \tag{6}$$

Here $C^2 \equiv R = Kg\beta Ah^2/\nu\chi$, and R is the analog of the Rayleigh number.

We shall henceforth consider only planar perturbations for which $v_y = 0$ and all quantities are independent of y. In this case it is convenient to introduce a stream function defined by the relations

$$v_x = \frac{\partial \Psi}{\partial z}, \quad v_z = -\frac{\partial \Psi}{\partial x} \tag{7}$$

Eliminating the pressure from Eqs. (5), we obtain

$$\begin{gathered} \Delta\Psi + C\left(\sin\alpha\frac{\partial T}{\partial z} + \cos\alpha\frac{\partial T}{\partial x}\right) = 0 \\ \Delta T - C\left(\sin\alpha\frac{\partial \Psi}{\partial z} + \cos\alpha\frac{\partial \Psi}{\partial x}\right) = 0 \end{gathered} \tag{8}$$

We consider perturbations periodic along the z axis:

$$\Psi = \varphi(x)\exp(ikz), \quad T = \theta(x)\exp(ikz)$$

The equations for the amplitudes of the perturbations $\varphi(x)$ and $\theta(x)$ assume the form

$$\begin{gathered} \varphi'' - k^2\varphi + C(ik\sin\alpha\theta + \cos\alpha\theta') = 0 \\ \theta'' - k^2\theta - C(ik\sin\alpha\varphi + \cos\alpha\varphi') = 0 \end{gathered} \tag{9}$$

The prime indicates differentiation with respect to x. The boundary conditions for the system (9) can be written in the following form:

$$\varphi = 0, \quad \theta = 0 \ \text{ or } \ \theta' = 0 \ \text{ for } \ x = \pm 1 \tag{10}$$

The system (9) with the boundary conditions (10) constitutes a characteristic value problem defining, for a given layer angle of inclination α and perturbation wave number k, a critical value C; upon reaching this critical value the equilibrium becomes unstable relative to perturbations with a given wavelength.

The general solution of the system (9) has the form

$$\varphi = \sum_{j=1}^{4} a_j \exp(iq_j x), \quad \theta = \sum_{j=1}^{4} b_j \exp(iq_j x) \tag{11}$$

$$(b_1 = ia_1, \ b_2 = ia_2, \ b_3 = -ia_3, \ b_4 = -ia_4)$$

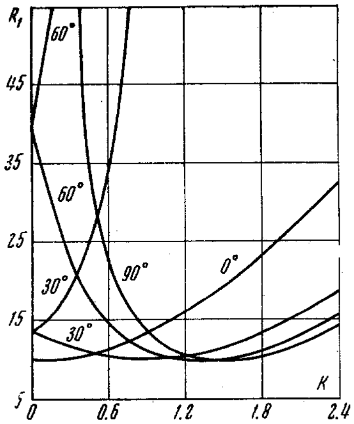

Fig. 1

where the q_j are the roots of the characteristic equation

$$q_1 = -L + M, \quad q_2 = -L - M, \quad q_3 = L + N, \quad q_4 = L - N$$
$$L = {}^1/_2\, C \cos\alpha, \quad M = {}^1/_2\, (C^2 \cos^2\alpha - 4kC \sin\alpha - 4k^2)^{1/2}$$
$$N = {}^1/_2\, (C^2 \cos^2\alpha + 4kC \sin\alpha - 4k^2)^{1/2}$$

Using the boundary conditions, we obtain a system of four linear homogeneous algebraic equations for the determination of the coefficients a_j. Setting the determinant of this system equal to zero, we obtain an equation connecting C, k, and α.

We consider first the case of ideally heat-conducting boundaries. We can then obtain the dependence of C on k and α in explicit form:

$$C = \pm (\cos\alpha)^{-2}\, [2k \sin\alpha \pm (4k^2 + n^2\pi^2 \cos^2\alpha)^{1/2}] \quad (n = 1, 2, 3, \ldots) \tag{12}$$

From Eqs. (12) it follows that the minimum critical Rayleigh number R_* is independent of the layer inclination angle:

$$R_* = \min (C^2) = n^2\pi^2$$

To it there corresponds the wave number

$$k_* = {}^1/_2\, n\pi \sin\alpha$$

The family of neutral curves $R_1(k)$ for n = 1 is shown in Fig. 1 for various angles α. Perturbations with k = 0 are the most dangerous only for a vertical layer. This conclusion differs qualitatively from the results obtained in the problem concerning the stability of an inclined layer of a viscous liquid where the transition to cellular perturbations occurs at a specific critical angle [4, 5].

We can also obtain an explicit expression for the stream function. For example, for the fundamental level n = 1 we have

$$\Psi = a \cos\, (\eta - Lx + kz) \cos {}^1/_2\, \pi x \tag{13}$$

where a and η are arbitrary numbers, corresponding to the fact that the solution is determined to within a normalization and a translation. It is evident from Eq. (13) that the convective cell boundaries on which $\psi = 0$ are straight lines. Their equations, for the most dangerous perturbations ($k = {}^1/_2\, \pi \sin\alpha$, $L = {}^1/_2\, \pi \cos \cdot \alpha$), have the form

$$x \cos\alpha - z \sin\alpha = 2\eta/\pi + S \quad (S = \pm 1, \pm 3, \pm 5, \ldots) \tag{14}$$

i.e., the cells are separated from one another by vertical straight lines, the distance between which along the horizontal is two or one layer thicknesses. By varying the parameter η we can displace this system of lines to the right or to the left, which corresponds to a translation along the z axis. The period of the translation is

$$l = 2\Delta\eta/\pi \sin\alpha \tag{15}$$

where $\Delta\eta$ denotes the change in η.

In the case of the vertical layer the arbitrariness of η leads to the result that we can have a solution with a single cell occupying the whole layer or we can have a solution with two cells. The boundary between them is parallel to the layer walls and can be situated at an arbitrary distance from the walls. Thus, for a vertical layer the critical number R is doubly degenerate.

We consider now the case of thermally insulated boundaries. Imposing on the general solution of the system (9) the corresponding boundary conditions, we obtain the equation

$$\begin{vmatrix} \exp(iq_1) & \exp(iq_2) & \exp(iq_3) & \exp(iq_4) \\ \exp(-iq_1) & \exp(-iq_2) & \exp(-iq_3) & \exp(-iq_4) \\ -q_1 \exp(iq_1) & -q_2 \exp(iq_2) & q_3 \exp(iq_3) & q_4 \exp(iq_4) \\ -q_1 \exp(-iq_1) & -q_2 \exp(-iq_2) & q_3 \exp(-iq_3) & q_4 \exp(-iq_4) \end{vmatrix} = 0 \tag{16}$$

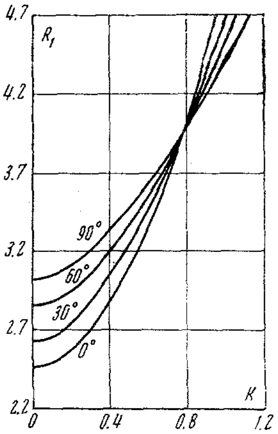

Fig. 2

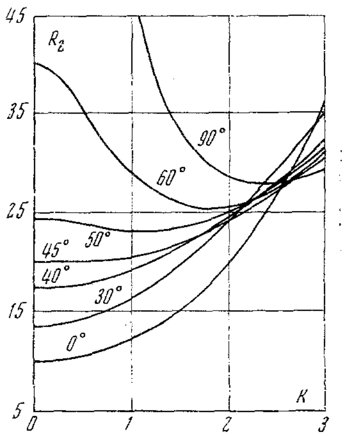

Fig. 3

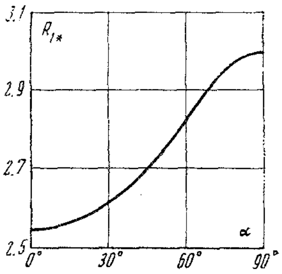

Fig. 4

Equation (16) defines the function $C(k, \alpha)$ implicitly. For small wave numbers (long wavelength perturbations) we write C as a power series expansion in k:

$$C = C_0 + C_1 k^2 + C_2 k^4 + \dots \tag{17}$$

The expansion must contain only even powers of the wave number since it is evident from the system (9) that C does not depend on the sign of k. The system of levels breaks up into two groups: for the one ("even" levels) $C_0 = n\pi/\cos\alpha$; for the other ("odd" levels) C_0 is determined from the equation

$$\operatorname{tg}(C_0 \cos\alpha)/C_0 \cos\alpha = 1/\sin^2\alpha \tag{18}$$

Calculations show that for the first odd level (the fundamental one) $C_1 > 0$. (The expression for C_1 is rather involved and will not be given here.)

Thus the most "dangerous" perturbations are those with $k = 0$.

For the second level (the first even level)

$$C_1 = (1 - 4\sin^4\alpha)/\pi\cos^3\alpha\ ; \tag{19}$$

from these it follows that $C_1 > 0$ for $\alpha < 45°$, and that $C_1 < 0$ for $\alpha > 45°$, and perturbations with finite k lead to instability. In the neighborhood of the critical value of the angle $\alpha_0 = 45°$, $C_1 = -4\pi^{-1} \cdot (\alpha - \alpha_0)$, and the wave number of the most dangerous perturbations depends on α according to the law

$$k_* = \sqrt{2/\pi C_2}\,(\alpha - \alpha_0)^{1/2} \tag{20}$$

where C_2 is taken for $\alpha = \alpha_0$.

To study the behavior of the neutral curves for finite k we solved Eq. (16) numerically.

Figure 2 shows the neutral curves $R_1(k)$ for the fundamental instability level for various layer orientations. For $k = 0$ they have a minimum for all α, and they increase monotonically as k increases. The neutral curves $R_2(k)$ for the second level are shown in Fig. 3. For $\alpha < 45°$ they have a minimum at $k = 0$. When $\alpha = 45°$ a change in the form of the instability occurs, and as the inclination angle is increased still further cellular perturbations lead to an equilibrium crisis.

In contrast to the case of heat-conducting boundaries, here the minimum critical numbers R_* depend on the layer orientation. This dependence, fairly weak for the fundamental level (Fig. 4), becomes significant at the higher levels. The number of extrema on the neutral curves R(k) increases for the higher levels and depends on the layer orientation.

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LITERATURE CITED

1. E. R. Lapwood, "Convection of a fluid in a porous medium," Proc. Cambr. Philos. Soc., 44, No. 3, 508-521 (1948).

2. Y. Kato and T. Masuoka, "Criterion for the onset of convective flow in a fluid in a porous medium," Intern. J. Heat Mass Transfer, 10, No. 3, 297-309 (1967).
3. V. S. Sorokin, "Variational methods in the theory of convection," Prikl. Matem. i Mekhan., 17, No. 1 (1953).
4. G. Z. Gershuni, E. M. Zhukhovitskii, and R. N. Rudakov, "On the theory of Rayleigh instability," Prikl. Matem. i Mekhan., 31, No. 5 (1967).
5. G. Z. Gershuni and E. M. Zhukhovitskii, "On the Rayleigh instability of a planar layer of a liquid with free boundaries," Uchen. Zapisk. Permsk. Univ., No. 184 (1968).